Fundamentals of Algebra, Geometry, and Trigonometry

(Self-Study Course)

This training is offered exclusively through the Pennsylvania Department of Transportation, Business Leadership Office, Technical Training and Development Section, located in Harrisburg, PA. For information about this training, contact, Mary Sharp at (717) 705-4170, marsharp@state.pa.us.

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Course Overview

Introduction

- ❖ Math plays a very large role in the design and construction of highways and bridges. Although many math problems can be solved with the help of a calculator or computer software, it is still important to understand the basic principles used to solve the problems. This enables the appropriate set-up, entry, and verification of the calculations. In addition, manual calculations may be necessary when a computer is not available.

- ❖ This course provides an explanation of concepts, step-by-step procedures, and illustrations to help participants:
  - Use the properties of numbers to solve the basic calculations required in the design and construction of highways and bridges.
  - Use basic equations to solve math problems.
  - Use coordinate planes and the principles of algebra to manipulate equations.
  - Apply geometric formulas, such as area and volume to solve basic math problems.
  - Apply trigonometric functions to solve problems involving triangles and angles.

Using This Course

- ❖ This is a self-study training course. The student is encouraged to find a mentor in the office to ask specific questions and to supplement the information found in this workbook.

- ❖ Since this is a self-study course, the amount of time it will take the student to complete each section will vary. It is important to take the time necessary to understand each concept before moving on to the next.

- ❖ Each part of the course builds on the information that has preceded it and prepares the student for information to follow. The following table outlines the main sections of the course and their suggested order of completion.

Course Sections in Their Suggested Order of Completion	Main Topics
1. Number Theory	• Integers • Absolute Value • Exponents • Computation Properties • Order of Operations
2. Algebra	• Variables and Expressions • Manipulating Equations • Exponents • Power Rules for Products and Quotients • Quotient Rule for Exponents
3. Coordinate Geometry	• Rectangular Coordinate Planes • Intercepts • Slope • Distance • Graphs on the Rectangular Coordinate Plane
4. Geometry	• Angles • Polygons • Triangles • Quadrilaterals • Circles
5. Trigonometry	• Trigonometric Ratios • Area of an Oblique Triangle • Law of Sines • Law of Cosines

Course Structure

- For each topic, there is an objective that explains what the student should know or be able to do after completing the instructional materials. In some cases, sample test items are given to provide additional explanation of the objective.

- Following the box that contains the objective, the topic will be explained.

   This symbol is used to indicate the definition of a new term.

- Once a topic has been explained, one or more examples are provided.

Additional Practice

- For additional practice, complete the problems in the file titled "Additional Exercises." A separate file titled "Additional Exercises Answer Key" contains the problem solutions.

Course Evaluation

- There is a course evaluation (i.e., test) to allow students to assess their level of understanding and progress. Once the test is complete, an answer key is provided for a self-check.

- For each question missed, the student should review the relevant materials in the workbook. If necessary, the student should also discuss the question with their mentor for additional explanation.

Number Theory

Introduction

What is Number Theory?

- **Number theory** is the branch of pure mathematics concerned with the properties of numbers in general, and integers in particular (integer examples: −2, −1, 0, 1, 2), as well as the wider classes of problems that arise from their study.

Why Study Number Theory?

- Many jobs within PennDOT, especially within the Bureau of Design, require a knowledge of mathematics. Number theory forms the foundation of most mathematical knowledge required at PennDOT, including algebra, coordinate geometry, geometry, and trigonometry.

- A knowledge of number theory makes it possible to solve the complex calculations required to design and build highways and bridges that are safe, functional, and reliable.

Introduction to Integers

Objective:

- Using the course materials as reference, and without the use of a calculator, be able to solve (write the answer to) addition, subtraction, multiplication and division problems that involve integers.

  Sample test items: $(-8)+7=$, $6-(-9)=$, $(-5)(-8)=$.

Integer Rules

Integer – integers are numbers that are positive, negative, or zero and can be written without a fractional or decimal component. (Examples: -2, -1, 0, 1, 2, 65, and -756 **are** integers; 1.6, 1½, -2.4, and -5/6 **are not** integers.)

A **fraction** is a number that can represent part of a whole. The top number (i.e. numerator) represents the number of parts of a whole, and the bottom number (i.e. denominator) indicates how many parts make up the whole. An example is 3/4, in which the numerator "3" tells us that the fraction represents 3 equal parts, and the denominator "4" tells us that 4 parts make up a whole. Therefore, 3/4 means we have 3 out of 4 parts.

Sign – the sign of a number indicates if the number is positive (+ sign), or negative (- sign). A positive number is greater than zero and a negative number is less than zero.

Rule Number 1 - Adding Integers

When adding like signs, add the numbers and keep the sign.

Examples:

a. $5 + 7 = 12$

b. $(-8) + (-5) = -13$

When adding unlike signs, take the difference (subtract) and keep the sign of the higher number.

Examples:

a. $(-9) + 7 = -2$

b. $5 + (-3) = 2$

Rule Number 2 - Subtracting Integers

In mathematics, the word **expression** is a term for any correctly written combination of numbers and/or mathematical symbols.

When subtracting integers you "add the opposite sign."

$(-3)-9=$ can be written as $(-3)+(-9)=$

Once the expression is written as an addition problem, follow addition rules.

$(-3)+(-9)=(-12)$

Example:

a. $(-15)-(6)=(-15)+(-6)=(-21)$

Rule Number 3 - Multiplying and Dividing Integers

When like signs are multiplied or divided, the answer is positive.

Examples:

a. $4 \times 9 = 36$

b. $12 \div 3 = 4$

c. $(-5)(-7) = 35$
note that $(-5)(-7)$ means $(-5) \times (-7)$

d. $(-24) \div (-6) = 4$

When unlike signs are multiplied or divided, the answer is negative.

Examples:

a. $45 \times (-2) = -90$

b. $(-15) \div 3 = -5$

c. $(12)(-10) = -120$

d. $24 \div (-3) = -8$

Absolute Value

Objective:

- Using the course materials as reference, and without the use of a calculator, be able to evaluate (write the answer to) numerical expressions that contain absolute values.

  Sample test items: $|3| =$, $|-7+3| =$.

Variable – a variable is a symbol that stands for a value that may vary. In our explanation of absolute value below, we will use the variable "x" to refer to any number.

The distance between a number x and 0 is called **absolute value**. The absolute value of x is written in symbols as $|x|$. Absolute value is the distance (magnitude) between x and 0 on a number line. For example, $|5| = 5$ and $|-5| = 5$ since both 5 and -5 are a distance of 5 units from 0 on a number line. **Hint:** the distance (absolute value) is always positive or zero; it is never negative.

Examples:

a. $|4| = 4$

b. $|-3| = 3$

c. $|0| = 0$

d. $|5+4| = 9$

e. $|-8+3| = 5$

Exponents (Powers)

Objective:

- Using the course materials as reference, and without the use of a calculator, be able to evaluate (write the answer to) numerical expressions that contain exponents (powers).

  Sample test items: $3^3 =$, $(\frac{1}{2})^2$.

Expressions with exponents

In mathematics, a **product** is the result of multiplying (for example: 8 is the product of 2×4), or a product is an expression that identifies factors to be multiplied (for example: the product of 2 and 3 is another way of saying 2×3).

In algebra, products frequently occur that contain repeated multiples of the same factor. For example, the volume of a cube whose sides each measure 2 cm is $2 \times 2 \times 2$ cubic cm. We may use **exponential notation** to write such products in a more compact form. For example $2 \times 2 \times 2$ may be written as 2^3.

The "2" in 2^3 is called the base. It is the repeated factor. The "3" in 2^3 is called the exponent and is the number of times the base is used as a factor. The expression 2^3 is called an exponential expression, $2^3 = 2 \times 2 \times 2 = 8$.

Additional exponent rules:

- Any number to the power of 1 is itself. Examples: $5^1 = 5$, $-10^1 = -10$.
- Any nonzero number to the power of 0 is 1. Examples: $5^0 = 1$, $-10^0 = 1$. Consider 0^0 to be a special case. In some areas of mathematics, 0^0 is defined as 1. In other areas of mathematics it is considered undefined.

Examples:

a. $3^2 = 3 \times 3 = 9$

b. $(-5)^3 = -5 \times -5 \times -5 = -125$

c. $2^4 = 2 \times 2 \times 2 \times 2 = 16$

d. $7^1 = 7$

e. $(\frac{2}{3})^2 = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

Properties

Objectives:

- Using the course materials as reference, when given a sample expression, be able to identify by name the commutative, associative, and distributive properties.

  Sample test item: Identify the property illustrated in the following problem, $10+5=5+10$.

- Using the course materials as reference, be able to rewrite an expression using the commutative, associative, and distributive properties.

  Sample test item: Using the distributive property, rewrite the following expression, $-(4-5)$.

There are a number of properties or rules that govern our computations.

Commutative Property of Addition or Multiplication	Adding (or multiplying) can be done in any order without changing the sum or product (i.e., doesn't change the answer). $4+3=3+4$ and $6\times 9=9\times 6$
Associative Property of Addition or Multiplication	Changing the grouping of three numbers will not change the sum (or product) (i.e., doesn't change the answer). $(17+9)+91=17+(9+91)$ and $2\times(3\times 6)=(2\times 3)\times 6$

Distributive Property of Multiplication over Addition	Multiplying the factor outside the parenthesis by each term (items separated by + or $-$) inside the parenthesis simplifies the problem. $7(3+8)=7\times 3+7\times 8$ and $-(3+4)=-3+(-4)$

Order of Operations

Objective:

- Using the course materials as reference, and without the use of a calculator, be able to simplify (write the answer to) numerical expressions using the proper order of operations.

Sample test items: $2\left[5+2(8-2)\right]=$, $\dfrac{16+\left|13-4\right|+4^2}{17-4}=$.

In mathematics, an **operation** is an action or procedure which produces a new value from one or more input values. (Addition, subtraction, multiplication, and division are examples of common math operations.)

As algebra problems become more complex, they frequently have parentheses, exponents, and square roots, and may involve more than one operation. In such cases, it is critical to perform the operations in the correct order. This is the correct order in which to simplify an expression:

❶ Perform all operations inside parentheses (or any other grouping symbol, such as brackets [ ] or braces { }), starting with the innermost set. Also, simplify the numerator and the denominator of a fraction separately.

❷ Simplify any expression with exponents or square roots. (Note that a square root of a number x is a number r such that $r^2 = x$. For example, 3 is the square root of 9 because $3^2 = 9$.)

❸ Multiply or divide, proceeding from left to right.

❹ Add or subtract, proceeding from left to right.

Examples:

a. $2\left[4+2(4-2)\right]$

$2\left[4+2(2)\right]$

$2\left[4+4\right]$

$2\left[8\right]$

16

b. $-5+3\times(-10)$

$-5+(-30)$

-35

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Algebra

What is Algebra?

Algebra is a branch of mathematics that uses mathematical statements to describe relationships between things that vary over time. These variables include things like the relationship between the supply of an object and its price. When using a mathematical statement to describe a relationship, letters are used to represent the quantity that varies, since it is not a fixed amount. As discussed in the Number Theory section, these letters are referred to as variables.

Why Study Algebra?

Algebra is used widely in all forms of engineering, including electrical, civil, chemical, and mechanical engineering. At PennDOT, algebra is used to determine many of the requirements for highway and bridge design. The following are just two of the many algebraic equations used at PennDOT.

Calculating stopping sight distance (SSD):

$$d = 1.47\,\mathrm{t}V + 1.075\frac{V^2}{a} \text{ (U.S. Customary)}$$

$$d = 0.278\,t\,V + 0.039\frac{V^2}{a} \text{ (Metric)}$$

Calculating the required superelevation of a horizontal curve:

$$e = \frac{V^2}{15R} - f \text{ (U.S. Customary)}$$

$$e = \frac{V^2}{127R} - f \text{ (Metric)}$$

Variables and Expressions

Objective:

- Using the course materials as reference, and a calculator for basic math calculations only, be able to evaluate (write the answer to) an algebraic expression.

  Sample test item: Find the value of $4a + 2b - 3c$, if $a = 2$, $b = 3$, and $c = 2$.

Definition of Variables and Expressions

As we learned in the Number Theory section, letters that represent numbers are called **variables** and a combination of letters and numbers is an **expression**. For example $3 + x$, $8y$, and $5x + 9$ are expressions. In these expressions, x and y are variables because they are used to represent numbers.

Evaluating Expressions

The value of an expression changes depending on the value of each variable. To find the value of an expression, replace the variables with their values.

> **Note:** In math, we often represent multiplication by using a dot or the $\times$ symbol $(2 \times 3 \text{ or } 2 \cdot 3)$. Either of these symbols is correct; however, it is more common in algebra to see the dot. In addition, in algebra we often write multiplication without the dot or $\times$ symbol. If there is no operation sign between two letters, or between a letter and a number, you assume it is multiplication. For example:

$$I = p \cdot r \cdot t \text{ is written as } I = prt$$

Examples:

a. Find the value of $6x - 2y$, if $x = 4$ and $y = 9$.

Replace x with 4 and replace y with 9. Then use the order of operations to find the answer.

$$\begin{gathered} 6x - 2y \\ 6(4) - 2(9) \\ 24 - 18 \\ 6 \end{gathered}$$

b. Evaluating in a Formula. Whenever we are evaluating in a formula, it is important to make sure our dimensions (units of measure) are compatible. For example, since cm is used for b, cm must also be used for h.

$\mathrm{A} = \frac{1}{2} \cdot b \cdot h$ is written $\mathrm{A} = \frac{1}{2} bh$

Find Area(A) when $b = 9$ cm and $h = 24$ cm

$$\mathrm{A} = \frac{1}{2} bh$$

$$\mathrm{A} = \frac{1}{2}(9)(24)$$

$$\mathrm{A} = 4.5(24) \text{ or } \frac{1}{2}(216)$$

$$\mathrm{A} = 108\,\mathrm{cm}^2 \quad (\text{note: } \mathrm{cm} \times \mathrm{cm} = \mathrm{cm}^2)$$

c. This formula is used for calculating stopping sight distance (SSD) (in metric measure). The formula is:

$$d = 0.278\, t\, V + 0.039 \frac{V^2}{a}$$

Solve given the following values:

$t = 1.6$ seconds (time to react)
$V = 80$ km/hr (initial speed of vehicle)
$a = 3.4$ m/s^2 (deceleration rate)
$d =$ the required stopping sight distance (m)

$$d = 0.278(1.6)(80) + 0.039 \frac{80^2}{3.4}$$

$$d = .278(1.6)(80) + \frac{249.6}{3.4}$$

$$d = 35.584 + 73.412$$

$$d = 108.996 \text{ m (or rounded to 2 decimal places is } 109.00 \text{ m)}$$

General Rule for Rounding. Round to 3 decimal places while doing calculations. Round the final answer to 2 decimal places.

Manipulating Equations in Algebra

Objective:

- Using the course materials as reference, and a calculator for basic math calculations only, be able to manipulate an equation in order to solve (write the answer to) for a specified variable.

  Sample test items: Given $5x-8=6x$, solve for x. Given $4a+8b-2c=d$, solve for c if $a=2, b=2,$ and $d=8$.

A **real number** is:

- Any number that can be shown on an infinitely long number line.
- Any positive or negative number, including zero, that can be expressed as a fraction.

An **equation** is a mathematical statement that two expressions are equal. For example, $5x-8=6x$ and $4a+8b-2c=d$ are equations.

A **linear equation** is an equation in which each term is either a constant or the product of a constant and (the first power of) a single variable. Linear equations can have one or more variables. When the solutions for a linear equation are graphed on a coordinate plane, they form a straight line.

To solve a linear equation for x, we write a series of simpler equations, all equivalent to the original equation, so that the final equation has the form

$x =$ a number, or, a number $= x$.

The first property of equality that helps us write simpler equivalent equations is the addition property of equality.

Addition Property of Equality

If *a*, *b*, and *c* are real numbers, then the following are equivalent equations:

$a = b$ and $a + c = b + c$

This property guarantees that adding the same number to both sides of an equation does not change the solution of the equation. Since subtraction is defined in terms of addition, we may also **subtract the same number from both sides** without changing the solution.

Examples:

a. Solve: $x - 4 = 15$

To solve for x, we first get x alone on one side of the equation. To do this, we add 4 to both sides of the equation.

$x - 4 = 15$	
$x - 4 + 4 = 15 + 4$	Add 4 to both sides
$x = 19$	Simplify

The solution is 19.

b. Solve: $y + 2.4 = 1.4$

To solve for y, we subtract 2.4 from both sides of the equation.

$y + 2.4 = 1.4$	
$y + 2.4 - 2.4 = 1.4 - 2.4$	Subtract 2.4 from both sides
$y = -1.0$	Simplify

The solution is -1.0.

c. Solve: $5x - 6 = 6x$

To solve for x, we subtract $5x$ from both sides of the equation.

$5x - 5x - 6 = 6x - 5x$	Subtract 5x from both sides
$-6 = 1x$	Simplify (note that to solve $6x - 5x,$ we subtract 5 from 6 to get 1, and we keep the variable x to get $1x$)
$x = -6$	

The solution is -6.

As useful as the addition property of equality is, it cannot help us solve every type of equation. For example, adding or subtracting a value on both sides of the equation does not help solve $3x = 24$. Instead, we apply another important property of equality, the **multiplication property of equality.**

Multiplication Property of Equality

If *a, b,* and *c* are real numbers and $c \neq 0$, then

$a = b$ and $ac = bc$

are equivalent equations.

This property guarantees that multiplying both sides of an equation by the same nonzero number does not change the solution of the equation. Since division is defined in terms of multiplication, we may also **divide both sides of the equation by the same nonzero number** without changing the solution.

Examples:

a. Solve: $5x = 100$

To solve for *x*, we first get *x* alone on one side of the equation. To do this, we divide by 5 on both sides of the equation.

$5x = 100$

$5x \div 5 = 100 \div 5$ Divide by 5 on both sides (note that $5x \div 5 = \frac{5x}{5} = 1x$)

$x = 20$ Simplify

The solution is 20.

b. Solve: $-6y = -36$

To solve for y, we divide by -6 on both sides of the equation.

$-6y = -36$	
$-6y \div -6 = -36 \div -6$	Divide by -6 on both sides
$y = 6$	Simplify

The solution is 6.

c. Solve: $\frac{x}{8} = 10$

To solve for x, we multiply both sides of the equation by 8.

$\frac{x}{8} = 10$	
$8 \cdot \frac{x}{8} = 10 \cdot 8$	Multiply both sides by 8
$x = 80$	Simplify

The solution is 80.

d. Solve: $\frac{x}{3.1} = 4.96$

To solve for x, we multiply both sides by 3.1.

$\frac{x}{3.1} = 4.96$	
$3.1 \cdot \frac{x}{3.1} = 4.96 \cdot 3.1$	Multiply both sides by 3.1
$x = 15.376$	Simplify

The solution is 15.376.

Using Both the Addition and Multiplication Properties

These equations use both the addition and the multiplication properties to solve.

Note: Always do the addition property first.

Examples:

a. Solve: $2x+5=25$

$2x+5=25$	
$2x+5-5=25-5$	Subtract 5 from both sides
$2x=20$	Simplify
$2x\div 2=20\div 2$	Divide both sides by 2
$x=10$	Simplify

The solution is 10.

b. Solve: $\frac{x}{3}-10=18$

$\frac{x}{3}-10=18$	
$\frac{x}{3}-10+10=18+10$	Add 10 to both sides
$\frac{x}{3}=28$	Simplify
$3\cdot\frac{x}{3}=28\cdot 3$	Multiply both sides by 3
$x=84$	Simplify

The solution is 84.

Solving Equations Can Be Combined with Evaluating

Examples:

a. Given $2L + 2W = P$, solve for L if $P = 40$, $W = 3$

$2L + 2W = P$	
$2L + 2(3) = 40$	Evaluate
$2L + 6 = 40$	Simplify
$2L + 6 - 6 = 40 - 6$	Subtract 6 from both sides
$2L = 34$	Simplify
$2L \div 2 = 34 \div 2$	Divide by 2 on both sides
$L = 17$	Simplify

The solution is 17.

b. $d = 0.278\, t\, V + 0.039 \dfrac{V^2}{a}$

We recognize this as the stopping sight distance equation that we used earlier to solve for d. We are now using the same equation, but manipulating it to solve for t.

Using the following values, solve the above equation for t.

$d = 156.4$ m

$\mathrm{V} = 80$ km/hr

$a = 3.4$ m/s^2

$156.4 = 0.278 \cdot t \cdot (80) + 0.039 \dfrac{(80)^2}{3.4}$	Evaluate
$156.4 = 22.24t + \dfrac{249.60}{3.4}$	Simplify
$156.4 = 22.24t + 73.412$	Subtract 73.412 from both sides
$82.988 = 22.24t$	Divide both sides by 22.24
$3.73\,\text{seconds} = t$	Simplify

The solution is 3.73 seconds.

Additional Properties in Algebra

Objective:

- Using the course materials as reference, be able to rewrite algebraic expressions using the commutative, associative, or distributive properties.

  Sample test item: Using the distributive property, rewrite the following expression, $3(a+c-d)$.

The laws of numbers apply to variables in the same way.

Commutative Property

$$x + y = y + x$$
$$xy = yx$$

Associative Property

$$(3 \cdot x) \cdot y = 3 \cdot (x \cdot y)$$
$$(3 + x) + y = 3 + (x + y)$$

Distributive Property

$$3(x + y) = 3 \cdot x + 3 \cdot y$$
$$-(x - y) = -x - (-y) = -x + y$$

Exponents

In the number theory section, we discussed how to evaluate an exponential expression.

$$3^2 = 3 \cdot 3 = 9$$

In this section, we will discuss how to add, subtract, multiply, and divide with exponents. We will also see how exponential expressions can themselves be raised to powers.

Using the Product Rule

Objective:

- Using the course materials as reference, and without the use of a calculator, be able to use the product rule to simplify (write the answer to) exponential expressions.

  Sample test items: Simplify $4^2 \cdot 4^4$. Simplify $x^3 \cdot x^4 \cdot x^2$.

By our definition of an exponent,

$$\begin{aligned} 4^3 \cdot 4^2 &= (4 \cdot 4 \cdot 4) \cdot (4 \cdot 4) \\ &= (4 \cdot 4 \cdot 4 \cdot 4 \cdot 4) \\ &= 4^5 \end{aligned}$$

Also,

$$\begin{aligned} x^2 \cdot x^4 &= (x \cdot x) \cdot (x \cdot x \cdot x \cdot x) \\ &= (x \cdot x \cdot x \cdot x \cdot x \cdot x) \\ &= x^6 \end{aligned}$$

In both cases, note that the result is the same as if we kept the base and added the exponents.

$$4^3 \cdot 4^2 = 4^{3+2} = 4^5 \quad \text{and} \quad x^2 \cdot x^4 = x^{2+4} = x^6$$

This suggests the following rule:

Product Rule for Exponents

If m and n are positive integers, and a is any number, then

$$a^m \cdot a^n = a^{m+n}$$

For example:

$$x^3 \cdot x^5 = x^{3+5} = x^8$$

In summary, to multiply two exponential expressions with the same base, we keep the base and add the exponents. We call this **simplifying** the expression.

Don't forget that if an exponent is not written, it is assumed to be one, $x = x^1$.

Examples:

Simplify

a. $2^3 \cdot 2^5 = 2^{3+5} = 2^8 = 256$

b. $y^2 \cdot y^5 = y^{2+5} = y^7$

Using the Power Rule

Objective:

- Using the course materials as reference, and without the use of a calculator, be able to use the power rule to simplify (write the answer to) exponential expressions.

  Sample test items: Simplify $(2^3)^2$. Simplify $(z^4)^5$.

Exponential expressions can be raised to powers.

For example:

$$(x^4)^2 = (x^4)(x^4)$$

$(x^4)^2$ means 2 factors of x^4 which can be simplified by the product rule as $x^4 \cdot x^4 = x^{4+4} = x^8$. Notice that the result would be the same if we multiplied the two exponents $(x^4)^2 = x^{4 \bullet 2} = x^8$.

This suggests the following rule:

Power Rule for Exponents

If m and n are positive integers, and a is any number, then,

$$(a^m)^n = a^{mn}$$

For example:

$$(y^4)^3 = y^{12}$$

In summary, to raise an exponential expression to a power, we keep the base and multiply the exponents.

Examples:

Simplify. Evaluate if possible.

a. $(4^2)^3 = 4^{2 \bullet 3} = 4^6 = 4096$

b. $\left(x^3\right)^3 = x^{3 \bullet 3} = x^9$

Using the Power Rules for Products and Quotients

Objective:

- Using the course materials as reference, and a calculator for basic math calculations only, be able to use the power rule to simplifying (write the answer to) products and quotients with exponential expressions.

  Sample test items: Simplify $(2a^2bc^4)^2$. Simplify $(\frac{a}{b})^8$.

When the base of an exponential expression is a product, the power rule still applies, but each factor of the product is raised to the power separately. For example:

$$(3x^2)^3 = 3^3 \cdot x^{2 \bullet 3} = 3^3 x^6 = 27x^6$$

Notice that to simplify the expression we raise each factor within the parentheses to a power of 3.

Therefore, we have the following rule:

Power of a Product Rule

If n is a positive integer, and a and b are any numbers, then

$$(ab)^n = a^n b^n$$

For example:

$$(2x^3)^4 = 2^4 x^{12} = 16x^{12}$$

In summary, to raise a product to a power, we raise each factor to the power.

Examples:

Simplify

a. $(xy)^2 = x^2 y^2$

b. $(2y)^3 = 2^3 y^3 = 8y^3$

In mathematics, a **quotient** is two quantities to be divided. An example quotient is $\frac{x}{y}$.

When we raise a quotient to a power, we simplify by raising both the numerator and the denominator to the specified power.

For example:

$$\left(\frac{x}{y}\right)^3 = \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} = \frac{x \cdot x \cdot x}{y \cdot y \cdot y} = \frac{x^3}{y^3}$$

In general, the following rule applies to all quotients:

Power of a Quotient Rule

If n is a positive integer, and a and c are any numbers, then

$$\left(\frac{a}{c}\right)^n = \frac{a^n}{c^n}$$

For example:

$$\left(\frac{x}{4}\right)^2 = \frac{x^2}{4^2} = \frac{x^2}{16}$$

Examples:

Simplify

a. $\left(\frac{s}{t}\right)^7 = \frac{s^7}{t^7}$

b. $\left(\frac{2x^3}{3y^5}\right)^4 = \frac{(2x^3)^4}{(3y^5)^4} = \frac{2^4 x^{3\bullet 4}}{3^4 y^{5\bullet 4}} = \frac{2^4 x^{12}}{3^4 y^{20}} = \frac{16x^{12}}{81y^{20}}$

Using the Quotient Rule for Exponents

Objective:

- Using the course materials as reference, and a calculator for basic math calculations only, be able to use the quotient rule for exponents to simplify (write the answer to) an algebraic expression.

  Sample test items: Simplify $\frac{3^4}{3^2}$. Simplify $\frac{9x^2y^4}{3xy^2}$.

Another way to simplify exponents involves quotients with the same base. For example:

$$
\begin{aligned}
\frac{x^4}{x^2} &= \frac{x \cdot x \cdot x \cdot x}{x \cdot x} \\
&= \frac{x \cdot \cancel{x} \cdot \cancel{x} \cdot x}{\cancel{x} \cdot \cancel{x}} \\
&= x \cdot x \\
&= x^2
\end{aligned}
$$

Notice the result is the same if we subtract exponents on the common bases.

$$\frac{x^4}{x^2} = x^{4-2} = x^2$$

The following rule states this for all quotients that have common bases.

Quotient Rule for Exponents

If m and n are positive integers, and a is any number, then

$$\frac{a^m}{a^n} = a^{m-n}$$

For example:

$$\frac{x^7}{x^3} = x^{7-3} = x^4$$

In summary, to divide one exponential expression by another with a common base, we keep the base and subtract the exponents.

Examples:

Simplify

a. $\frac{y^7}{y^2} = y^{7-2} = y^5$

Zero Exponent

Objective:

- Using the course materials as reference, and without the use of a calculator, be able to simplify (write the answer to) an exponential expression containing a zero exponent.

  Sample test items: $(-50)^0 =$, $(4a^5y^{10})^0 =$.

Let's now give meaning to the expression x^0. We can do this using the quotient rule as well as using what we already know about fractions.

$$\frac{x^2}{x^2} = x^{2-2} = x^0$$

or

$$\frac{x^2}{x^2} = \frac{x \cdot x}{x \cdot x} = 1$$

Since $\frac{x^2}{x^2} = x^0$ and $\frac{x^2}{x^2} = 1$, we define that $x^0 = 1$ as long as $x \neq 0$.

$$\boxed{a^0 = 1, \text{ as long as } a \text{ is not } 0.}$$

For example:

$4^0 = 1$

Example:

Simplify

a. $x^0 = 1$

Negative Exponents

Objective:

- Using the course materials as reference, and without the use of a calculator, be able to simplify (write the answer to) an exponential expression using a negative exponent.

  Sample test items: $2a^{-2} =$, $\dfrac{x^{-2}}{z^{-4}} =$.

Using the quotient rule, we can get an understanding of negative exponents.

$$\frac{x^4}{x^2} = x^{4-2} = x^2 \quad \text{also,} \quad \frac{x^3}{x^7} = x^{3-7} = x^{-4}$$

Using our knowledge of fractions, we see that

$$\frac{x^3}{x^7} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{x \cdot x \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x} = \frac{1}{x \cdot x \cdot x \cdot x} = \frac{1}{x^4}$$

Since $\dfrac{x^3}{x^7} = x^{-4}$ and $\dfrac{x^3}{x^7} = \dfrac{1}{x^4}$, we can state that $x^{-4} = \dfrac{1}{x^4}$

From this example, we state the definition of negative exponents.

For Negative Exponents:

If a is any number other then 0, and n is an integer, then

$$a^{-n} = \frac{1}{a^n}$$

For example:

$$x^{-3} = \frac{1}{x^3}$$

In summary, another way to write a^{-n} is to take its reciprocal and change the sign of its exponents. (Note, the reciprocal of $\frac{1}{3}$ is $\frac{3}{1}$ or 3, and the reciprocal of 3 or $\frac{3}{1}$ is $\frac{1}{3}$.)

Examples:

Simplify

a. $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

b. $2x^{-3} = 2 \cdot \frac{1}{x^3} = \frac{2}{1} \cdot \frac{1}{x^3} = \frac{2}{x^3}$

Using Fractional Exponents

Objective

- Using the course materials as reference, and a calculator for basic math calculations only, be able to simplify (write the answer to) an exponential expression using a fractional exponent.

  Sample test item: Evaluate $a^{\frac{2}{3}}$, when $a = 4$. Evaluate $8^{\frac{1}{3}} =$.

How to Rewrite Fractional Exponents

For any nonzero real number b, and any integer m and n, with $n > 1$,

$b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$ except when $b < 0$

Examples:

Simplify

a. $4^{\frac{1}{2}} = \sqrt[2]{4^1} = 2$ *

b. $x^{\frac{2}{3}} = \sqrt[3]{x^2}$ **

c. $x^{\frac{1}{2}} = \sqrt[2]{x^1}$ *

d. $x^{\frac{2}{3}} = \sqrt[3]{x^2}$ **

e. $16^{\frac{3}{4}} = \sqrt[4]{16^3}$ **

* Note that a base raised to the $\frac{1}{2}$ power is the same as taking the square root of a number. The square root symbol $\sqrt{}$ is an operation symbol that asks us what number times itself will give us the number under the symbol. Example: $\sqrt{4} = 4^{\frac{1}{2}} = 2$ because $2 \cdot 2 = 4$ and $\sqrt{9} = 9^{\frac{1}{2}} = 3$ because $3 \cdot 3 = 9$.
** $\sqrt[3]{}$ is the cube root or $\frac{1}{3}$ power and $\sqrt[4]{}$ is the fourth root or $\frac{1}{4}$ power. Therefore $\sqrt[3]{8} = 8^{\frac{1}{3}} = 2$ because $2 \cdot 2 \cdot 2 = 8$ and $\sqrt[4]{16} = 16^{\frac{1}{4}} = 2$ because $2 \cdot 2 \cdot 2 \cdot 2 = 16$.

We can also use fractional exponents to evaluate an algebraic expression or formula.

Examples:

Evaluate

a. $x^{\frac{1}{2}}$, when $x = 9$

$9^{\frac{1}{2}} = \sqrt{9} = 3$

b. $y^{\frac{1}{2}}$, when $y = 5$

$5^{\frac{1}{2}} = \sqrt{5} = 2.236$ (note that finding the square root of 5 requires a calculator and 2.236 is a rounded number)

Example:

a. Manning's Equation – an equation used to compute the velocity of water flowing in an open channel is listed below:

$$V = \frac{1.49}{n} R^{\frac{2}{3}} S^{\frac{1}{2}}$$

Where V is velocity in ft/sec, n is the roughness coefficient of the channel, R is the hydraulic radius in ft, S is the slope of the channel.

Using the following values, $n = .012$, $R = 1.3\,\text{ft}$, $V = 25\,\text{ft/sec.}$ Solve the above equation for S.

$$V = \frac{1.49}{n} R^{\frac{2}{3}} S^{\frac{1}{2}}$$

$$25\,\text{ft} = \frac{1.49}{.012}(1.3\,\text{ft})^{\frac{2}{3}} S^{\frac{1}{2}}$$

$$25\,\text{ft} = (124.167)(1.191\,\text{ft})S^{\frac{1}{2}}$$

$$25\,\text{ft} = (147.883\,\text{ft})S^{\frac{1}{2}}$$

$$\frac{25\,\text{ft}}{147.883\,\text{ft}} = S^{\frac{1}{2}}$$

$$.169 = S^{\frac{1}{2}} = \sqrt{S}$$

$$(.169)^2 = (\sqrt{S})^2$$

$$.029 = S \text{ (since this is slope, it could be considered } \frac{\text{ft}}{\text{ft}})$$

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Coordinate Geometry

What is Coordinate Geometry?

Coordinate geometry is the study of geometry using the principles of algebra. In coordinate geometry, a coordinate plane is often used to manipulate equations for planes, straight lines, and squares, often in two and sometimes in three dimensions of measurement.

Why Study Coordinate Geometry?

When designing and constructing roadways and bridges, many concepts from coordinate geometry are used. These can include:

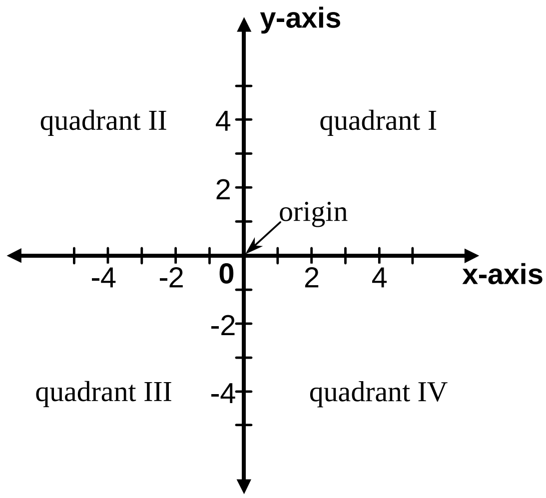
- Determining the distance between two points.
- Finding the midpoint and slope of a line.
- Determining if lines are parallel or perpendicular.
- Finding the area and perimeter of a shape as defined by its points on a plane.
- Defining the equations of curves and circles.

Plotting Points in the Rectangular Coordinate Plane

Objective:

- Using the course materials as reference, and given a blank rectangular coordinate plane, be able to plot ordered pairs on the rectangular coordinate plane.

Ordered pairs are used to describe the location of a point in a plane. Begin with a horizontal and a vertical axis. Each axis is a number line and the axes intersect at the 0 coordinate of both. This point of intersection is called the origin. These two number lines (or axes) divide the plane into four regions called quadrants. Each quadrant is numbered with Roman numerals.

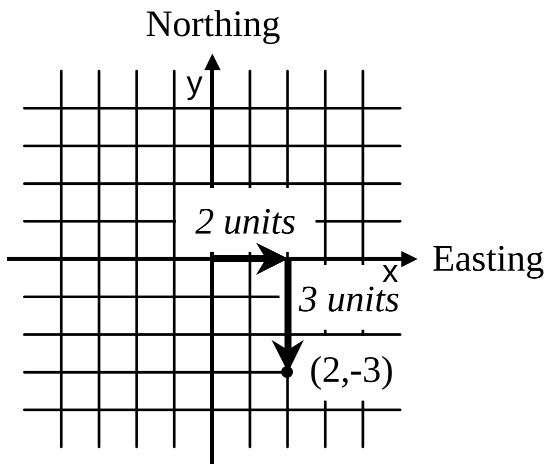


We call this a rectangular coordinate system or plane.

A point is a position on the plane. It has no size or dimension. An ordered pair of numbers (x, y) notes the position. We list the horizontal, or x-coordinate, first and the vertical, or y-coordinate, second. For the ordered pair $(2,-3)$, $x = 2$ and $y = -3$.

In highway design, the y axis is often referred to as the **Northing** direction and the x axis as the **Easting** direction. Note, in highway design we normally list the Northing (y-coordinate) before the Easting (x-coordinate). For example: Northing $= -3$, Easting $= 2$, or $y = -3$ and $x = 2$.

To plot or graph the point corresponding to the ordered pair (b, c), we start at the origin. We then move b units to the right or left (right if b is positive, left if b is negative). From there, we move c units up or down (up if c is positive, down if c is negative). For example, to plot the point corresponding to the ordered pair (2, -3), we start at the origin, move 2 units to the right, and from there move 3 units down.



Ordered Pairs

Objective:

- Using the course materials as reference, be able to find (write the value of) the corresponding x or y value: when given a linear equation and an x or y value.

An equation with one variable, such as $x+4=5,$ has only one solution, which is 1 (the number 1 is the value of the variable x that makes the equation true).

An equation with two variables, such as $2x+y=8$, has solutions for two variables, one for x and one for y. For example, $x=3$ and $y=2$ is a solution of $2x+y=8$ because if x is replaced with 3 and y is replaced with 2, we get a true statement.

$$\begin{aligned} 2x+y&=8 \\ 2(3)+2&=8 \\ 8&=8 \end{aligned}$$

The solution $x=3$ and $y=2$ can be written as (3,2), an ordered pair of numbers. In general, an ordered pair is a solution of an equation with two variables if replacing the variables by the values results in a true statement. For example, another ordered pair solution of $2x+y=8$ is (4,0). Replacing x with 4 and y with 0 results in a true statement.

$$\begin{aligned} 2x+y&=8 \\ 2(4)+0&=8 \\ 8&=8 \end{aligned}$$

Completing Ordered Pairs

When given a linear equation (an equation whose graph is a straight line) and a value for x or y, it is possible to find the unknown value of x or y in the ordered pair.

Example:

Complete each ordered pair so that it is a solution to the equation $3x + y = 12$.
a. (0,_) b. (_,6)

a. In the ordered pair (0,_), the x value is 0.
We let $x = 0$ in the equation and solve for y.

$$\begin{aligned} 3x + y &= 12 \\ 3(0) + y &= 12 \\ 0 + y &= 12 \\ y &= 12 \end{aligned}$$

b. In the ordered pair (_,6), the y value is 6. We let $y = 6$ in the equation and solve for x.

$$\begin{aligned} 3x + y &= 12 \\ 3x + 6 &= 12 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

Solutions of equations with two variables can also be recorded in a table of paired values like this.

x	y

Example:

a. Complete the table for the equation $2x + y = 8$.

If we substitute 1, 3, and 4 in for x, we determine that the following ordered pairs exist: (1,6), (3,2), and (4,0). Placed in the table, the ordered pairs appear as follows,

x	y
1	6
3	2
4	0

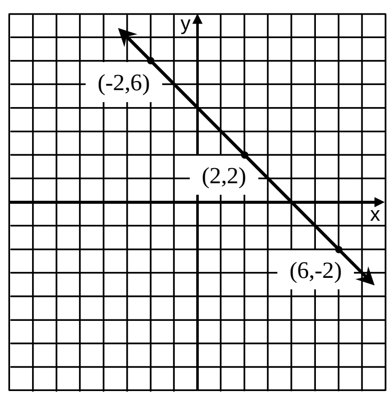
Graphing Linear Equations

Objective:

- Using the course materials as reference, be able to graph a linear equation on a blank rectangular coordinate plane by calculating and plotting ordered pairs.

  Sample test item: By calculating and plotting ordered pairs, graph the linear equation $2x + y = 6$.

We know that equations with two variables may have more than one solution. For example, $x + y = 4$ has (2,2) and (1,3) as solutions. In fact, this equation has an infinite number of solutions. For example, other solutions are $(6,-2)$ and $(-2,6)$. If we graphed any three of these points, they would form a straight line.



☞ A linear equation can be written in the form $Ax + By = C$, where A, B, C are numbers and A and B are not both 0. The graph of a linear equation is a straight line.

A straight line is determined by just two points. Therefore, to graph a linear equation with two variables, we need to find just two of its infinite number of solutions. Once we do this, we can plot the solution points and draw the line connecting the points. Usually, we find a third point to check.

In highway design, we describe roadway geometrics in terms of straight lines (called tangents) and curves. We use coordinate geometry to precisely locate points that define a roadway alignment, such as the beginning or end of a curve, or the point of intersection of two lines.

Example:

a. Graph the linear equation $2x + y = 5$.

To graph this equation, we find three ordered pair solutions of $2x + y = 5$. To do this, we choose a value for one variable, x or y, and solve for the other variable. For example, if we let $x = 1$, then $2x + y = 5$ becomes:

$$\begin{aligned} 2x + y &= 5 \\ 2(1) + y &= 5 \\ 2 + y &= 5 \\ y &= 3 \end{aligned}$$

Since $y = 3$ when $x = 1$, the ordered pair (1,3) is a solution.
Next, we let $x = 0$.

$$\begin{aligned} 2x + y &= 5 \\ 2(0) + y &= 5 \\ 0 + y &= 5 \\ y &= 5 \end{aligned}$$

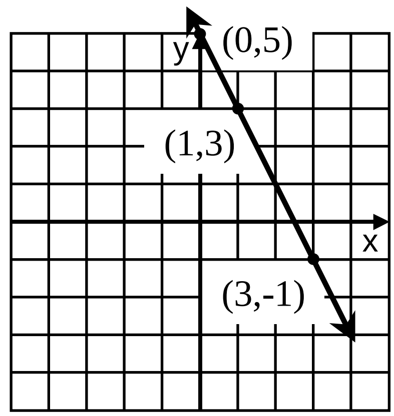
The ordered pair (0,5) is a second solution.

The two solutions found so far allow us to draw the straight line that is the graph of all solutions of $2x + y = 5$. However, we will find a third ordered pair as a check. Let $y = -1$.

$$\begin{aligned} 2x + y &= 5 \\ 2x + (-1) &= 5 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

The third solution is (3,-1). These three ordered pair solutions are listed in the table and plotted on the coordinate plane. The graph of $2x + y = 5$ is the line through the three points.

x	y
0	5
1	3
3	-1

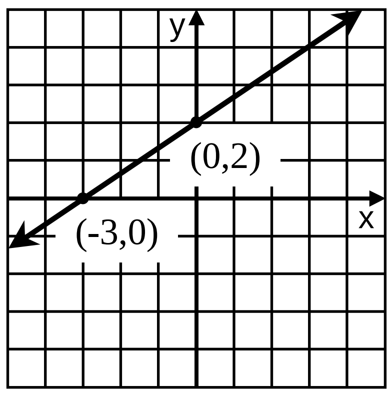


Identifying Intercepts

Objective:

- Using the course materials as reference, be able to identify (write the values of) the x and y intercepts when given a linear equation.

The graph of $y = \frac{2}{3}x + 2$ is shown below. Notice that this graph crosses the y axis at the point (0,2). This point is called the **y intercept**. Likewise, the graph crosses the x axis at $(-3,0)$. This point is called the **x intercept**.



Notice that if y is 0, the corresponding x value is the x intercept. Likewise, if x is 0, the corresponding y value is the y intercept.

Finding x and y Intercepts

- ❑ To find the x intercept, let $y = 0$ and solve for x.
- ❑ To find the y intercept, let $x = 0$ and solve for y.

Example:

a. Find the x and y intercepts of $y = 4x - 8$.

We let $y = 0$ to find the x intercept and $x = 0$ to find the y intercept.

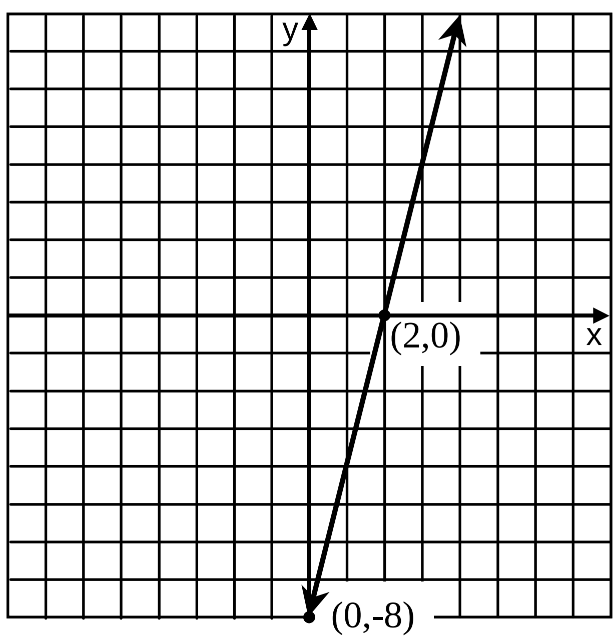
Let $y = 0$.

$$y = 4x - 8$$
$$0 = 4x - 8$$
$$8 = 4x$$
$$2 = x$$

Let $x = 0$.

$$y = 4x - 8$$
$$y = 4(0) - 8$$
$$y = 0 - 8$$
$$y = -8$$

Therefore, the point (2,0) and $(0,-8)$ are the intercepts of $y = 4x - 8$. We can actually plot these two points, connect the two points, and have the graph of $y = 4x - 8$. This is another way to graph a line.



Thus, another way to graph a linear equation (straight line) is to use the x and y intercepts of the line.

Graphing Vertical and Horizontal Lines

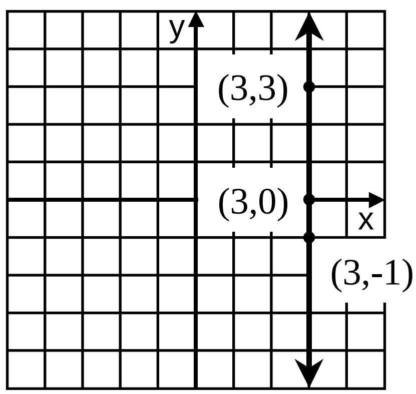
Objective:

- Using the course materials as reference, be able to graph an equation that can be represented by a vertical or horizontal line.

Vertical Lines

The equation $x = 3$, for example, is a linear equation with two variables because it can be written in the form $x + 0y = 3$. The graph of this is a vertical line as shown below.

For any y value chosen, notice that x is 3. No other value for x satisfies $x + 0y = 3$. We will use the ordered pair solutions (3,3), (3,0), $(3,-1)$ to graph $x = 3$.

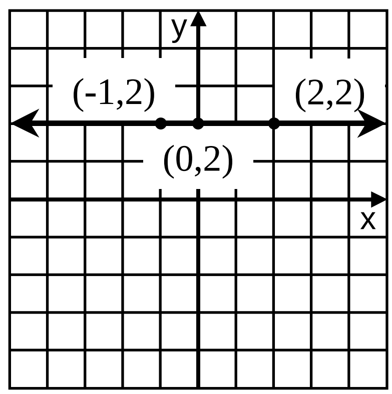


The graph is a vertical line with an x intercept of 3. Note that this graph has no y intercept because x is never 0.

Horizontal Lines

The equation $y = 2,$ for example, is a linear equation with two variables because it can be written in the form $0x + y = 2.$ The graph of this is a horizontal line as shown below.

For any x value chosen, notice that y is 2. No other value for y satisfies $0x + y = 2.$ We will use the ordered pair solutions $(2,2), (0,2), (-1,2)$ to graph $y = 2.$



The graph is a horizontal line with a y intercept at 2 and no x intercept because y is never 0.

Summary

In general, we have the following:

Vertical Lines

The graph of $x = c,$ where c is any number, is a vertical line with an x intercept of $c.$

Horizontal Lines

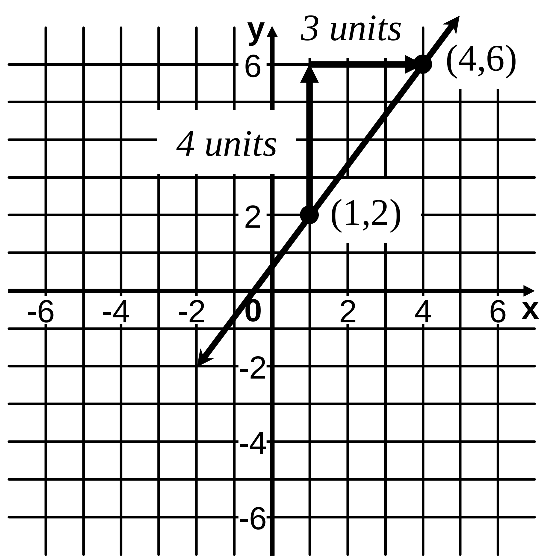
The graph of $y = c,$ where c is any number, is a horizontal line with a y intercept of $c.$

Defining Slope

Objective: • Using the course materials as reference, and by using the slope formula, be able to find (write the value of) the slope of a straight line if given two points on the line. Be able to graph the line.

In math, the slant or steepness of a line is formally known as its **slope**. We measure the slope of a line by the ratio of vertical change (rise) to the corresponding horizontal change (run) as we move along the line from left to right.

On the line below, for example, suppose we begin at the point (1,2) and move to the point (4,6). The vertical change is the change in y coordinates: $6 - 2 = 4$ or 4 units. The corresponding horizontal change is the change in x coordinates: $4 - 1 = 3$ or 3 units.



The slope of this line then, is $\frac{4}{3}$. This means that for every 4 units of change in y coordinates, there is a corresponding change of 3 units in x coordinates. It makes no difference what two points of a line are chosen to find its slope. The slope of the line is the same everywhere on the line.

The slope m of the line containing the points $(x_{1,} y_1)$ and (x_2, y_2) is given by:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in y}}{\text{change in x}} = \frac{y_2 - y_1}{x_2 - x_1}$$

A percent grade is the same as its slope given as a percent. An 8% grade means that for every rise of 8 units there is a run of 100 units.

$$8\%\ \text{grade} = \frac{\text{rise of 8}}{\text{run of 100}} = 0.08\ \text{(expressed as a decimal)}$$

In highway design we often talk about the slopes of pipes, pavement cross slopes, and the cut and fill slopes on roadway cross sections. Pipe slopes are typically given in percentages. A -3% slope indicates a drop of 3 units vertically for every 100 units horizontally. Likewise, pavement cross slopes are also given in percentages. In urban areas, the maximum pavement cross slope (or superelevation) is 6%. In rural areas the maximum cross slope can be as high as 8%.

A slope written as 2:1 means for every 2 feet (horizontally/run) the slope will rise 1 foot (vertically/rise). We read this in math as a slope of $\frac{1\,\text{rise}}{2\,\text{run}}$.

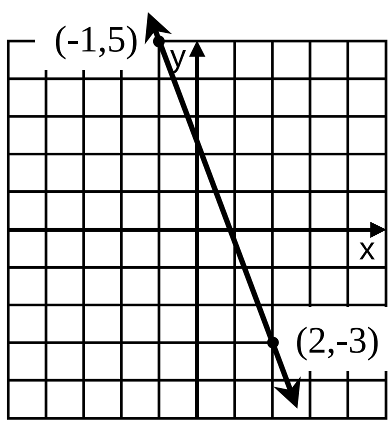
Example:

a. Find the slope of the line through $(-1,5)$ and $(2,-3)$. Graph the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-5}{2-(-1)} = \frac{-8}{3}$$

The slope of the line is $\frac{-8}{3}$.

It doesn't matter which point is (x_1, y_1) or which is (x_2, y_2). However, once an x coordinate is called x_1, make sure its corresponding y coordinate is called y_1.



Finding the Slope of a Line Given Its Equation

Objective:

- Using the course materials as reference, be able to find (write the value of) the slope given a linear equation.

If a linear equation is solved for y, the coefficient of x is its slope. (The coefficient is the number before x in the x term. For example, in the term $2x$, the coefficient is 2). In other words, the slope of the line given by $y = mx + b$ is m, which is the coefficient of x.

Example:

a. Find the slope of $5x + 4y = 10$.

First solve for y.

$$4y = -5x + 10$$
$$y = \frac{-5}{4}x + \frac{10}{4}$$

When the equation is solved for y, the coefficient of x is $\frac{-5}{4}$. Therefore, the slope is $\frac{-5}{4}$.

Slopes of Common Lines

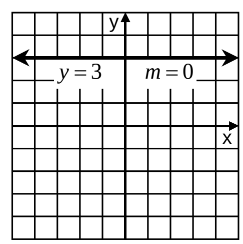
Horizontal and Vertical Lines

Objective:

- Using the course materials as reference, be able to find (write the value of) the slope of a vertical or horizontal line if given the appropriate x or y intercept.

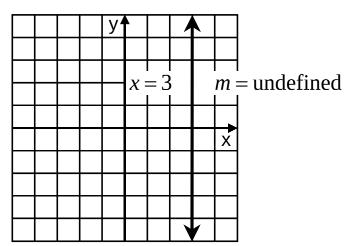
A horizontal line is always expressed in the form $y = a\ number$. Since the y values of this line are always the same for any x value, as discussed previously, the difference between them is zero. Therefore,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0$$



A vertical line is always $x = a\ number$. Since the x values of this line are always the same, the difference between them is zero. Therefore,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{0} = \text{undefined}$$



All horizontal lines have a slope of 0. All vertical lines have a slope that is undefined or no slope. (Note that a number divided by zero has no meaning; it is undefined.)

Parallel and Perpendicular Lines

Objective:

- Using the course materials as reference, be able to determine (write the answer to) whether two lines are parallel or perpendicular if given the linear equation for each line. The slope of each line should be calculated from the linear equation. The relationship between the slopes of each line should be used to determine whether the two lines are parallel or perpendicular.

Parallel Lines

Two lines in the same plane are **parallel** if they do not intersect.

Slopes of lines can help us determine whether lines are parallel. Since parallel lines have the same slant or steepness, they have the same slope.

$$y = 3x + 5$$
$$y = 3x - 2$$

Both of these lines have a slope of 3. Therefore, they are parallel to each other.

Perpendicular Lines

Reciprocal fractions are fractions where the numerator and denominator are switched. For example, $\frac{4}{3}$ is the reciprocal of $\frac{3}{4}$.

Two lines are **perpendicular** if they lie in the same plane and intersect at a 90 degree angle (right angle).

The slopes of perpendicular lines are negative reciprocals of each other.

For example:

$$\mathrm{y} = \frac{2}{3}x + 3$$
$$y = \frac{-3}{2}x + 1$$

Since the slopes $\frac{2}{3}$ and $\frac{-3}{2}$ are negative reciprocals, the two lines are perpendicular.

Example:

a. Determine whether the following pair of lines are parallel, perpendicular, or neither.

$$x + y = 5$$
$$2x + y = 5$$

Since we need to compare slopes, put the equation into the slope intercept form first (slope intercept form is $y = mx + b$).

$y = -x + 5$ and
$y = -2x + 5$

Now compare -1 and -2. They are neither the same nor negative reciprocals. Therefore, the lines are neither parallel nor perpendicular.

More Equations of Lines

Objective:

- Using the course materials as reference, when given a linear equation, be able to use the slope intercept form to graph the linear equation.

We know that when a linear equation is solved for y, the coefficient of x is the slope of the line. For example, the slope of the line whose equation is $y = 3x + 1$ is 3. The 1 in this equation is the y intercept.

Remember that the y intercept is where the line intercepts the y axis. The x value of the y intercept is always 0.

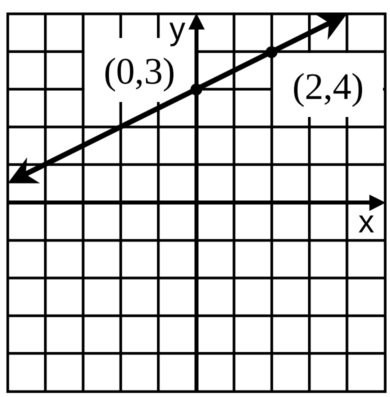
The **Slope Intercept Form** of a line is the equation of a line written as $y = mx + b$ where *m* is the slope of the line and *b* is the y intercept of the line.

Example:

a. Graph $y = \frac{1}{2}x + 3$.

When a linear equation with two variables is written in slope intercept form, $y = mx + b$, then m is the slope of the line and b is the y intercept of the line.

Since the equation $y = \frac{1}{2}x + 3$ is written in slope intercept form $y = mx + b$, the slope is $\frac{1}{2}$ and the y intercept is (0,3). To graph this equation, we first graph the y intercept at (0,3). From this point, we can find another point of the graph by using the slope $\frac{1}{2}$, recalling that slope is $\frac{\text{rise}}{\text{run}}$. We start at the intercept point and move 1 unit up and 2 units to the right. We stop at the point (2,4). The line through (0,3) and (2,4) is the graph of $y = \frac{1}{2}x + 3$.



Writing an Equation Given Slope and a Point

Objective:

- Using the course materials as reference, be able to graph a line given the slope of the line and a point on the line.

We already learned how to graph a line given its slope and y intercept. Here's an equation to use when given the slope and any point on the line. It is called **Point Slope Form**. This form is based on the slope formula.

$$y - y_1 = m(x - x_1)$$

Example:

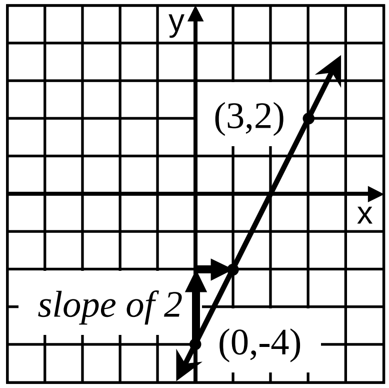
a. Find the equation of the line that has a slope of 2 and passes through the point (3,2). Graph the line.

   Substitute the given information into the Point Slope Form equation, $m = 2$, $x_1 = 3$, and $y_1 = 2$.

$$\begin{aligned} y - 2 &= 2(x - 3) \\ y - 2 &= 2x - 6 \\ y &= 2x - 4 \end{aligned}$$

To graph the line, use the two points that we know; (3,2) which was given, and $(0,-4)$ from the equation, and then use the slope to find a third point.

x	y
0	-4
1	-2
3	2



Point of Intersection of Two Lines

Objective:

- Using the course materials as reference, be able to find (write the value of) the point of intersection if given two linear equations.

The point of intersection (x,y) of two lines is:

$$x = \frac{B_2C_1 - B_1C_2}{A_2B_1 - A_1B_2}$$

$$y = \frac{A_1C_2 - A_2C_1}{A_2B_1 - A_1B_2}$$

where $A, B,$ and C are from the Standard Form of a line $Ax + By = C$ rearranged to read $Ax + By - C = 0$.

The **Standard Form** of a line is in the form Ax + By = C where A is a positive integer, and B, and C are integers.

Example:

a. Find the intersection of $x + y = -1$ and $-x + 4y = -4$.

Rearrange these two equations so that they equal zero: $x + y + 1 = 0$ and $-x + 4y + 4 = 0$.

Using the formulas:

$$x = \frac{B_2C_1 - B_1C_2}{A_2B_1 - A_1B_2}$$

$$y = \frac{A_1C_2 - A_2C_1}{A_2B_1 - A_1B_2}$$

if we let: $A_1 = 1$ $\quad B_1 = 1$ $\quad C_1 = 1$
$A_2 = -1$ $\quad B_2 = 4$ $\quad C_2 = 4$

$$x = \frac{(4\cdot 1) - (1\cdot 4)}{(-1\cdot 1) - (1\cdot 4)} = 0 \text{ and } y = \frac{(1\cdot 4) - (-1\cdot 1)}{(-1\cdot 1) - (1\cdot 4)} = -1$$

Therefore, the intersection of the two lines is the point $(0,-1)$.

Distance Between Two Points

Objective:
• Using the course materials as reference, be able to find (write the value of) the distance of a line between two given points (ordered pairs).

The distance (d) between any two points (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This formula is derived from the Pythagorean Theorem, which will be discussed in another section.

Example:

a. Suppose we want to find the distance between two points $(-1,8)$ and $(3,5)$. We can use the distance formula with $(x_1, y_1) = (-1,8)$ and $(x_2, y_2) = (3,5)$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

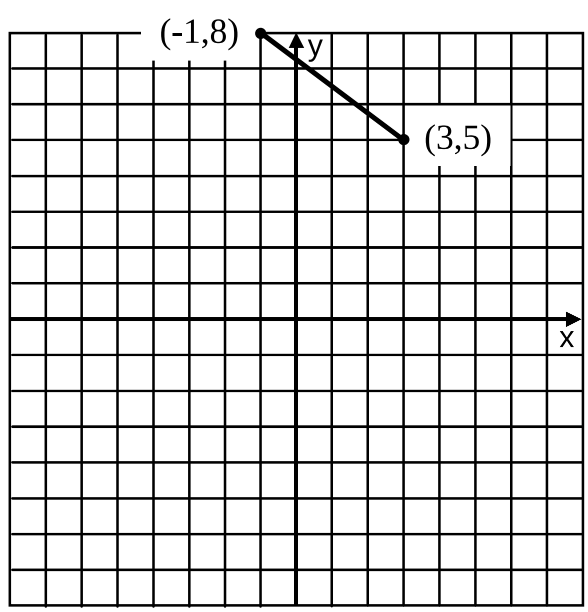
$$d = \sqrt{(3-(-1))^2 + (5-8)^2}$$

$$d = \sqrt{(4)^2 + (-3)^2}$$

$$d = \sqrt{16+9}$$

$$d = \sqrt{25}$$

$$d = 5$$



Other Graphs

Objective:

- Using the course materials as reference, be able to graph a quadratic equation (parabola) by calculating and plotting points.

A **quadratic equation** is an equation in which the highest power of an unknown quantity is a square. An example is $x^2 + 3x + 4 = 0$.

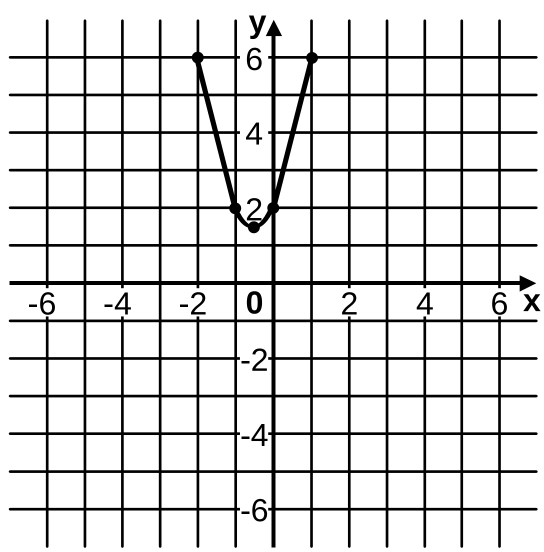
We just discussed graphs of the form $y = mx + b$ in two variables. The graph of a quadratic equation $y = ax^2 + bx + c$ is a shape called a parabola. The figure below shows the general shape of a parabola (all parabolas are vaguely "U" shaped and the highest or lowest point is called the vertex). In highway design, parabolic curves are used to define the elevation along vertical curves in a roadway profile.

We can graph any equation with x and y by picking values for x and solving for y, or by evaluating the equation for a specified value.

Example:

a. Graph $y = 2x^2 + 2x + 2$.

x	y
-2	6
-1	2
-0.5	1.5
0	2
1	6



Graphing Circles

Objective:

- Using the course materials as reference, be able to identify (write the value of) the radius and center of a circle, as well as graph the circle in a coordinate plane when given the equation of the circle.

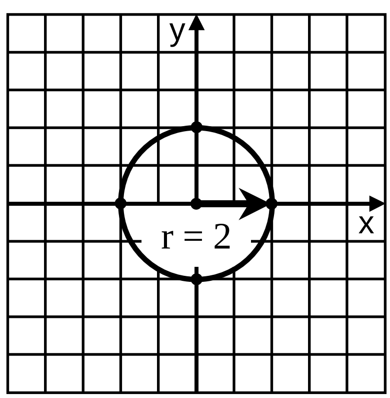
The **radius** of a circle is the length of the line from the center to any point on its edge.

$x^2 + y^2 = r^2$ is the equation for a circle with a center at the origin. In this equation, r is the radius of the circle.

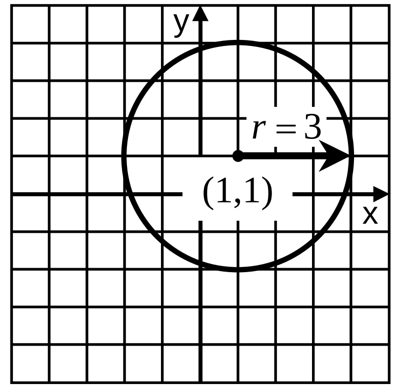
$(x-h)^2 + (y-k)^2 = r^2$ is the equation for a circle with a center at (h,k). In this equation, r is the radius of the circle.

Examples:

a. Graph $x^2 + y^2 = 4$. This circle has a center at (0,0) and a radius of 2. The graph looks like this:



b. Graph $(x-1)^2+(y-1)^2=9$.



Based on the equation $(x-h)^2+(y-k)^2=r^2$; we know that the center of the circle is (h,k) or $(1,1)$, and the radius of the circle is r or $\sqrt{9}=3$.

Geometry

What is Geometry?

Geometry means "earth measures." It is a part of mathematics that is concerned with questions of size, shape, and the relative position of figures. Geometry originated as a practical science dealing with surveying, measurements, areas, and volumes.

Geometry is widely used in the fields of science, engineering, and computers. It deals with the formulas for lengths, areas, and volumes. These formulas can include the circumference and area of a circle, area of a triangle, volume of a cylinder, sphere, and pyramid.

Why Study Geometry?

In highway and bridge design and construction, geometry plays a crucial role in everything from the initial land survey, to design and construction, to the calculation of contractor payments.

- During site surveys the construction area is represented by various geometric elements such as points, lines, arcs, circles, and other defined geometric shapes. The tools and calculations used by surveyors require various angle theorems (theories and equations) to find the location of features such as roadway alignments, utilities, and site boundaries.
- Geometric elements such as angles, arcs, and radii are used in designing roadways and bridges.
- Calculations of area, volume, and perimeter are often required to determine how much a contractor should be paid for construction work.

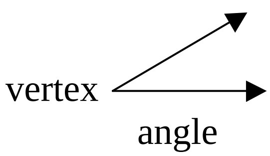
Geometry Definitions

Objectives:

- Using the course materials as reference, be able to:
  - Identify (write the name of) angles based on specific characteristics.
  - Calculate (write the value of) the unknown angle when given the diagram of two angles and information about the relationship between them.

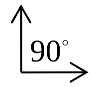
Ray: a straight line that originates at a point and extends in one direction from that point.

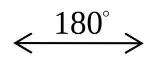
Angle: the union of two rays that share one and only one point; the endpoint of the rays. This point is also known as the vertex of the angle.



Please note, the angles and figures in the Geometry section may not be drawn to scale.

Common Angles

- A **right** angle measures exactly 90 degrees. 

- An **acute** angle is an angle that measures less than 90 degrees.
- An **obtuse** angle is an angle that measures more than 90 degrees.
- A **straight** angle is an angle that measures exactly 180 degrees. 

- **Complementary** angles are two angles whose measures total 90 degrees.
- **Supplementary** angles are two angles whose measures total 180 degrees.
- **Linear** angles are a pair of adjacent angles whose sum is a straight angle (180 degrees – straight line).

Measuring Angles

Objective:

- Using the course materials as reference, be able to convert (write the value of) DMS (degree, minute, seconds) into decimal degrees and decimal degrees into DMS.

Angles can be measured in degrees. Degrees can be broken down into even smaller units called minutes and seconds.

Complete Circle = 360 degrees
1 degree = 60 minutes
1 minute = 60 seconds

PennDOT right-of-way and construction plans specify the bearing (or direction) of roadway baselines in terms of their deflection angle from due north or south. Deflection angles are written using degrees, minutes, and seconds. Example: $N30^\circ 15' 20.22'' E$.

Angles measured in degrees, minutes, and seconds (DMS) need to be converted to decimal degrees for calculations.

Examples:

a. 46° (degrees), $29'$ (minutes), $15''$ (seconds) can be converted to a decimal form.

$$46 + \frac{29}{60} + \frac{15}{60^2 \text{ or } 3600} = 46 + .4833 + .0042 = 46.4875^\circ$$

b. Convert 12.3712° to DMS.

In 12.3712, 12 is the degree portion.
$.3712 \cdot 60 = 22.272$

In 22.272, 22 is the minute portion.
$.272 \cdot 60 = 16.32$

16.32 is the second portion.

Therefore, the answer is $12^\circ 22' 16.32''$

Angles in Parallel Lines

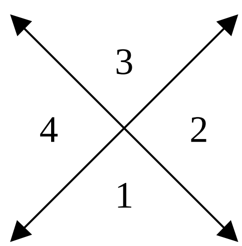
Objectives:

- Using the course materials as reference:
  - If given the measure of at least one angle in a vertical angle, be able to calculate (write the value of) an unknown angle.
  - If given two parallel lines and a transversal, be able to identify (write the answer to) which angles formed by the transversal are equal.
  - If given two parallel lines and a transversal, be able to identify (write the value of) an unknown angle if given the measure of at least one angle formed by the transversal.

Two lines are **parallel** if they lie in the same plane, but never meet.

Intersecting lines meet or cross at one point.

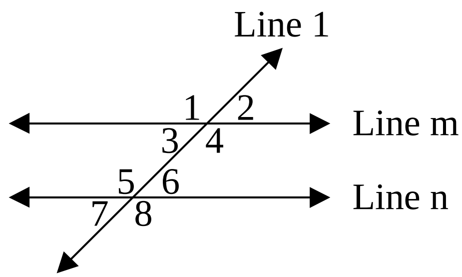
Two intersecting lines form **vertical angles**. Below, angles 1 and 3 are vertical angles. Angles 2 and 4 are also vertical angles. Vertical angles have equal measures. (Note, in this section *m* will mean the "measure of".)



$$m\angle 1 = m\angle 3$$
$$m\angle 2 = m\angle 4$$

Adjacent angles have the same vertex and share a side. Above, angles 1 and 2 are adjacent angles. Other pairs of adjacent angles are angles 2 and 3, angles 3 and 4, and angles 4 and 1.

A **transversal** is a line that intersects two or more lines in the same plane. Line l is a transversal that intersects lines m and n. The eight angles formed are numbered and certain pairs of these angles are given special names.



In the above diagram:

- **Corresponding angles** are: $\angle 1$ and $\angle 5$, $\angle 3$ and $\angle 7$, $\angle 2$ and $\angle 6$, $\angle 4$ and $\angle 8$
- **Exterior angles** are: $\angle 1$, $\angle 2$, $\angle 7$, and $\angle 8$
- **Interior angles** are: $\angle 3$, $\angle 4$, $\angle 5$, and $\angle 6$
- **Alternate interior** angles are: $\angle 3$ and $\angle 6$, $\angle 4$ and $\angle 5$

If two parallel lines are cut by a transversal, then the following angles in the above diagram are equal.

$\angle 1 = \angle 5 = \angle 4 = \angle 8$

$\angle 3 = \angle 7 = \angle 2 = \angle 6$.

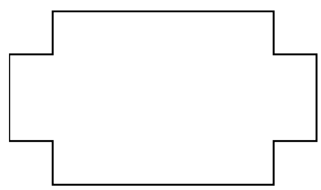
It can also be stated that corresponding angles are equal and alternate interior angles are equal.

Polygons

Objectives:

- Using the course materials as reference, be able to:
  - Calculate (write the value of) the sum of the interior angles if given the diagram of a polygon.
  - Calculate (write the value of) the sum of the interior angles if given the number of sides of a polygon.

A **polygon** is a closed figure in a plane that is made up of 3 or more sides.



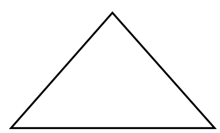
The sum of the interior angles of a polygon is equal to (n-2)180, where n is the number of sides of the polygon. Note: for all polygons, it is important to have all sides (or heights) in the same unit of measure. For example, if one side is measured in inches, then all sides should be measured in (or converted to) inches.

Triangles

Objectives:

- Using the course materials as reference, be able to:
  - Find (write the value of) the third angle of a triangle given the other two angles.
  - Find (write the value of) the unknown side given the other two sides of a right triangle.
  - Find (write the value of) the area of a triangle given its height and length.
  - Find (write the value of) the perimeter of a triangle given the length of its sides.

A **triangle** is a polygon with three sides.



There are several different types of triangles based on information about the sides or angles of the triangle.

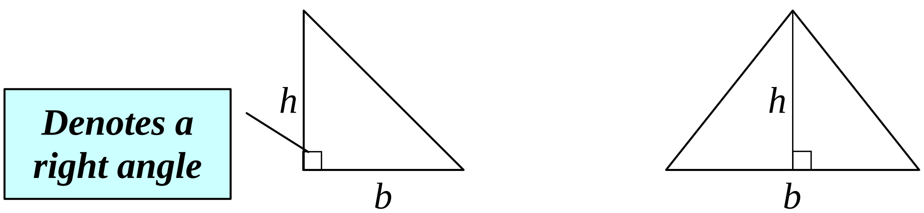
- **Right Triangle** is a triangle with a right angle (90 degrees).
- **Isosceles Triangle** is a triangle with two equal sides.
- **Equilateral Triangle** is a triangle with all sides equal.
- **Equiangular Triangle** is a triangle with all angles equal.
- **Oblique Triangle** is a triangle with no right angle.
- **Scalene Triangle** is a triangle with no sides equal.

Important Facts about Triangles

- The sum of the measures of the interior angles of a triangle equals 180 degrees. (This follows the general formula for polygons $(n-2)180$ where n is the number of sides of the polygon: $(3-2)180 = 180^{\circ}$.)

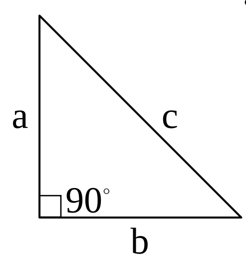
Area is the amount of space occupied by a two-dimensional shape or object. Note that when calculating area, the units of measure for the answer are squared. For example: if a triangle has a base of 4 inches and a height of 4 inches, the area (see area formula below) would be ½ of 4 inches times 4 inches, or ½ of 16 square inches (square inches can also be written in^2) = 8 in^2. Be sure to use the same units for all measurements. You cannot multiply feet times inches, it doesn't make a square measurement.

- The area of a triangle equals $\frac{1}{2}bh$ where b is the base and h is the height.



Perimeter is the length of the outer boundary of a two-dimensional shape or object.

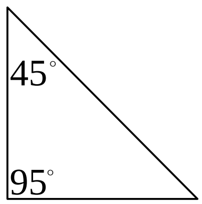
- The perimeter of a triangle equals the sum of the lengths of each side. In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the two legs. In a right triangle, the side opposite the right angle is called the hypotenuse (c) and the other sides are called legs (a and b).



Therefore, $a^2 + b^2 = c^2$. This is called the **Pythagorean Theorem** and it works for all right triangles.

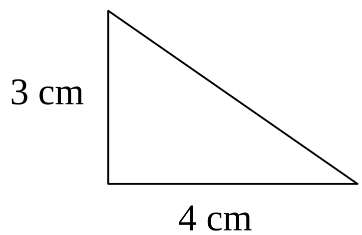
Examples:

a. Find the measure of the third angle of the triangle shown.



Solution: The sum of the measures of the angles of a triangle is 180 degrees. Since one angle measures 45 and the other angle measures 95 degrees, the third angle measures $180^\circ - 45^\circ - 95^\circ = 40^\circ$.

b. Find the length of the hypotenuse of a right triangle whose legs have lengths of 3 cm and 4 cm.



Because we have a right triangle, we use the Pythagorean Theorem. The legs are 3 cm and 4 cm, so let $a = 3$ and let $b = 4$ in the formula.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= c^2 \\ 9 + 16 &= c^2 \\ 25 &= c^2 \\ 5\,\text{cm} &= c \end{aligned}$$

Therefore, the hypotenuse is 5 cm.

c. Find the area of a triangle whose base is 10 in. and whose height is 4 in. Find the perimeter if the three sides are 10 in., 5 in., 7 in.

$$A = \frac{1}{2}bh$$
$$A = \frac{1}{2}(10)(4)$$
$$A = 20\,\text{in.}^2$$

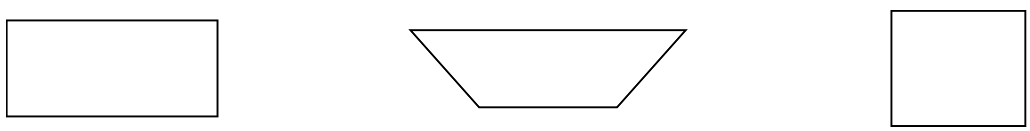
$$P = s_1 + s_2 + s_3$$
$$P = 10 + 5 + 7$$
$$P = 22\,\text{in.}$$

Quadrilaterals

Objective:

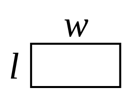
- Using the course materials as reference, be able to calculate (write the value of) the area or perimeter of a square, rectangle, or trapezoid if given the diagram of (or information about) a square, rectangle, or trapezoid.

Quadrilaterals are four-sided polygons.



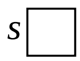
Three Common Quadrilaterals

A **rectangle** has four sides and four right angles. Its opposite sides are parallel and equal. The area of a rectangle is its length multiplied by its width. The perimeter of a rectangle is twice its length plus twice its width.

For a rectangle: 

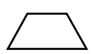
Area = length × width. This is usually written as $(A = lw)$.
Perimeter = 2(length) + 2(width). This is usually written as $(P = 2l + 2w)$.

A **square** is a rectangle with all sides equal. Therefore, the area of a square is the same as the area of a rectangle. Instead of calling the sides of the square length and width, we simply call them sides because all four sides have the same measurement. The perimeter of a square is 4 multiplied by the length of its side.

For a square: 

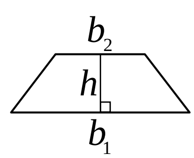
Area $=$ side $\times$ side <u>or</u> (side^2). This is usually written as $A = s^2$.
Perimeter $=$ 4(side). This is usually written as $P = 4s$.

A **trapezoid** has four sides of which only two are parallel. The parallel sides are called the bases. The area of a trapezoid is one-half the height multiplied by the sum of the bases. The height of a trapezoid is the distance between the two parallel sides (bases). The perimeter of a trapezoid is the sum of the lengths of the sides.

For a trapezoid: 

$\text{Area} = \frac{1}{2}h(b_1 + b_2)$

$\text{Perimeter} = s_1 + s_2 + s_3 + s_4$



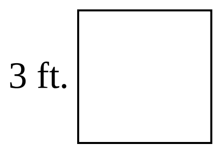
It does not matter which base you assign as b_1 and b_2.

Examples:

a. Find the area of a rectangle that is 6 meters long and 5 meters high.

$A = lw$
$A = (6)(5)$
$A = 30\text{ m}^2$

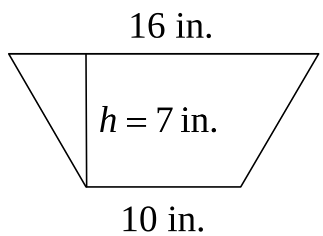
b. Find the area and perimeter of the following square.



$$A = s^2$$
$$A = 3^2$$
$$A = 9 \text{ ft.}^2$$

$$P = s + s + s + s$$
$$P = 3 + 3 + 3 + 3$$
$$P = 12 \text{ ft.}$$

c. Find the area of the following trapezoid.



$$A = \frac{1}{2} h(b_1 + b_2)$$
$$A = \frac{1}{2}(7)(16 + 10)$$
$$A = 3.5(26)$$
$$A = 91 \text{ in.}^2$$

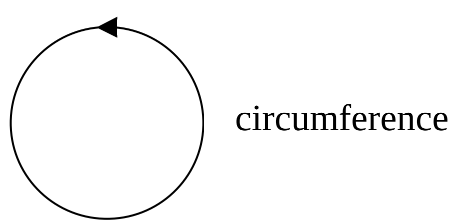
Circles

Objectives:

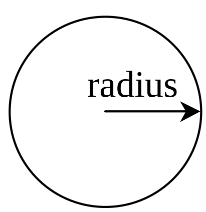
- Using the course materials as reference, be able to:
  - Find (write the value of) the radius and diameter of a circle when given the diagram of (or information about) a circle.
  - Find (write the value of) the circumference and area of a circle when given the radius of a circle.
  - Calculate (write the value of) the circumference and area of a circle when given the diameter of a circle.
  - Calculate (write the value of) the area of a semicircle or quarter circle when given the radius or diameter of a semicircle or quarter circle.

A **circle** is a closed plane curve of which every point is equally distant from a center point.

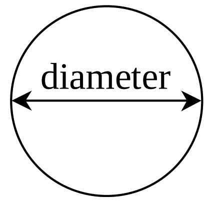
The **circumference** (C) of the circle is the outside boundary line of the circle.



The **radius** (r) is a straight line from the center of a circle to the circumference.



The **diameter** is the distance across the circle as you pass through the center.



The diameter $(d) = 2$ radii (r), or $d = 2r$.

Circumference (C) and Area (A) of a circle can be found by using the following formulas:

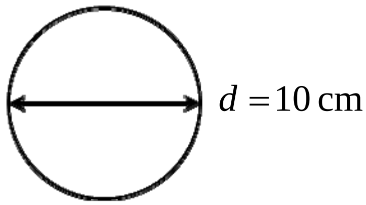
Circumference $(C) = \pi d$ <u>or</u> $2\pi r$
Area $(A) = \pi r^2$

A **semicircle** is a half circle. Since the area of a whole circle equals (πr^2), the area of a semicircle equals $(\frac{1}{2}\pi r^2)$.

π is a constant that we use when working with circles. It is the ratio of the circumference of a circle to the diameter. It is denoted by π which is the Greek letter *pi* (pronounced pie). It is approximately 3.14159265359…, but we use 3.14 for most calculations.

Examples:

a. Find the area and circumference of the following circle.



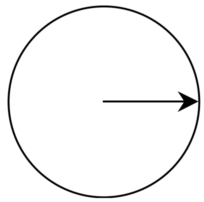
If $d = 10$ cm, then $r = 5$ cm

$A = \pi r^2$
$A = (3.14)(5)^2$
$A = 78.5\ \text{cm}^2$

$C = \pi d$ or $C = 2\pi r$. Since we are given the diameter, we use $C = \pi d$.
$C = (3.14)(10)$
$C = 31.4$ cm

Note: If you solve this problem using the π key on your calculator, you might get a slightly different answer than if you solve this problem using 3.14 for π. This is because the π key on most calculators will carry π out to more than 2 decimal places.

b. Find the area and circumference of the following circle if $r = 6$ ft.



$$A = \pi r^2$$
$$A = (3.14)(6)^2$$
$$A = (3.14)(36)$$
$$A = 113.04 \text{ ft.}^2$$

$$C = 2\pi r$$
$$C = 2(3.14)(6)$$
$$C = 37.68 \text{ ft.}$$

c. Find the area of a semicircle that has a radius of 4 in.

$$A = \frac{1}{2}\pi \text{ r}^2$$
$$A = \frac{1}{2}(3.14)\,(4)^2$$
$$A = \frac{1}{2}(3.14)16$$
$$A = 25.12 \text{ in.}^2$$

Unique Polygons

Objective:

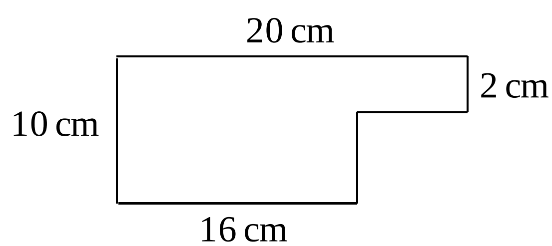
- Using the course materials as reference, be able to find the area and perimeter of unique polygons and other composite figures when given the diagram of (or information about) the unique polygon or composite figure.

To find the **area** of a unique polygon, try to break the total area into pieces that are squares, triangles, rectangles, trapezoids, or circles. Find the area of each piece, then add the areas of the individual pieces together.

To find the **perimeter** of a unique polygon, add the length of all the sides.

Example:

a. Find the area and perimeter of the following unique polygon.



Answer:

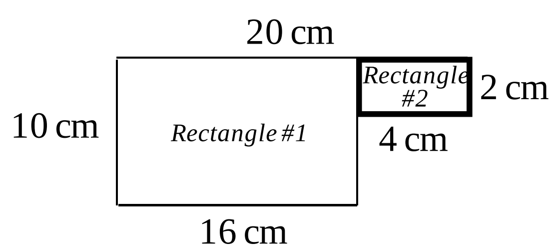
$A_1 = l\,w$ $\quad\quad$ $A_2 = l\,w$

$A_1 = 16(10)$ $\quad\quad$ $A_2 = 4(2)$

$A_1 = 160\text{ cm}^2$ $\quad\quad$ $A_2 = 8\text{ cm}^2$

$A_1 + A_2 = 160 + 8$

$\text{Total area} = 168\text{ cm}^2$



$$\text{Perimeter } = 10 + 20 + 2 + 4 + 8 + 16 = 60 \text{ cm}$$

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Trigonometry

What is Trigonometry?

Trigonometry is a branch of mathematics that studies how the sides and angles of a triangle are related to each other. For a given triangle, trigonometry allows for the unknown measurement of an angle or length of a side to be calculated given the known measurements of the other angles or sides.

Why Study Trigonometry?

There are an enormous number of applications for trigonometry and trigonometric functions. For instance, the technique of trigonometric triangulation is used in satellite navigation systems and in geography to measure distances between landmarks.

Fields that make use of trigonometry or trigonometric functions include land surveying and civil engineering.

- Trigonometry is used anywhere angles or curves are involved. Therefore, trigonometry is used in surveying to measure distances and heights, and it is used in highway and bridge design for calculating curves, elevations, angles, and distances.
- Trigonometry is used extensively in the design of roadway horizontal alignments.

Definitions

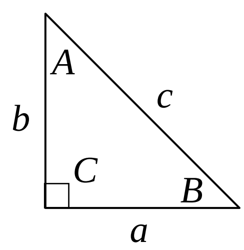
Objective:

- Using the course materials as reference, for any given right triangle, be able to calculate (write the value of) the sine, cosine, or tangent of any specified angle.

Please note, the angles and figures in the Trigonometry section may not be drawn to scale.

For an acute angle (an angle that measures less than 90 degrees) in a right triangle (a triangle with one right angle), we will consider three ratios involving the lengths of pairs of sides of the triangle.

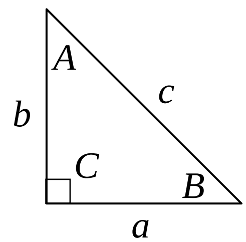
Each of these ratios is constant with respect to an acute angle of fixed measure. (Note that *A*, *B*, *C* represent the measure of each angle and *a*, *b*, *c* represent the measure of each side of the following triangle.)



$$\text{sine } A = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}} = \frac{a}{c}$$

$$\text{cosine } A = \frac{\text{length of side adjacent to } \angle A}{\text{length of hypotenuse}} = \frac{b}{c}$$

$$\tan A = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A} = \frac{a}{b}$$



S O H C A H T O A

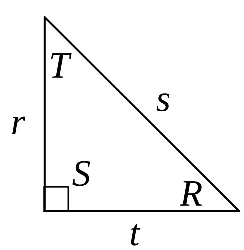
SOHCAHTOA is a handy way to remember the trigonometric relationships.
(SOH is sin, opp, hyp; CAH is cos, adj, hyp; TOA is tan, opp, adj.)

$$\sin\angle = \frac{\text{opp}}{\text{hyp}} \qquad \cos\angle = \frac{\text{adj}}{\text{hyp}} \qquad \tan\angle = \frac{\text{opp}}{\text{adj}}$$

$$\sin\angle A = \frac{a}{c} \qquad \cos\angle A = \frac{b}{c} \qquad \tan\angle A = \frac{a}{b}$$

Examples:

a. Given right triangle *RST*, write in terms of *r*, *s*, and *t* the three trigonometric ratios for $\angle R$.

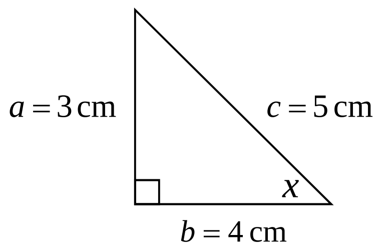


$$\sin\angle R = \frac{r}{s}$$

$$\cos\angle R = \frac{t}{s}$$

$$\tan\angle R = \frac{r}{t}$$

b. Calculate the sin x, cos x, tan x for the given right triangle.



$$\sin x = \frac{3}{5}\,\text{cm}$$
$$\cos x = \frac{4}{5}\,\text{cm}$$
$$\tan x = \frac{3}{4}\,\text{cm}$$

Using the Trigonometric Ratios to Find Unknown Measures

Objective:

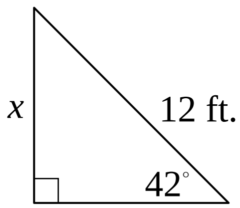
- Using the course materials as reference, for any given right triangle, be able to find (write the value of) the length of a side, or the measure of an acute angle using trigonometric ratios (sine, cosine, tangent, $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$).

To find the **length of a side** of a right triangle given the measure of one acute angle and the measure of one side:

1. Identify the placement of the sides of the right triangle with respect to the given angle.
2. Choose the trigonometric ratio.
3. Substitute the given values.
4. Substitute the value of the trigonometric ratio.
5. Solve.
6. Round.

Example:

a. Find, to the nearest tenth, the length of the side whose measure is represented by x.



$$\sin \angle = \frac{\text{opp}}{\text{hyp}}$$

$$\sin 42^\circ = \frac{x}{12}$$

$$.6691^* = \frac{x}{12}$$

$$\frac{.6691}{1} = \frac{x}{12}$$

$$1x = 12(.6691)$$

$$x = 8.029$$

$$x = 8.0 \text{ ft.}$$

* The value of trigonometric functions can be obtained from a calculator or a trigonometric table. The values from each will always be the same. For example, sin 30 degrees is always .5. Trigonometric values are usually rounded to four decimal places.

To find the **measure of an acute angle** of a right triangle given the length of two sides:

1. Identify the placement of the sides of the triangle.

2. Choose the trigonometric ratio.

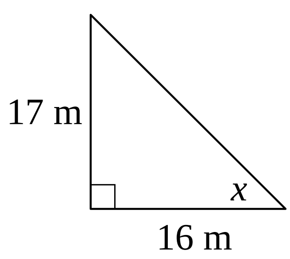
3. Substitute the given values.

4. Write the ratio as a decimal.

5. Use a calculator (or table) to find the approximate measure of the angle.

Example:

a. Find the measure of angle x to the nearest degree.



$$\tan\angle = \frac{\text{opp}}{\text{adj}}$$

$$\tan x = \frac{17}{16}$$

$\tan x = 1.0625$ (if we know the ratio of the opposite over adjacent, and need to find the angle, use inverse tangent ($\tan^{-1}$)).

$$\tan^{-1} \text{ of } 1.0625 = 46.735$$

$$\angle x = 47^\circ$$

Oblique Triangles

Objectives:

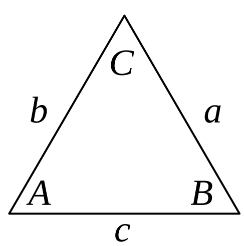
- Using the course materials as reference:
  - Be able to solve (write the value of) the area of an oblique triangle using the sine function.
  - When given an oblique triangle, be able to calculate (write the value of) the length of an unknown side or the value of an unknown angle using the law of sines or the law of cosines.

The trigonometric functions and the Pythagorean Theorem cannot be used to solve oblique triangles because oblique triangles do not contain a right angle. However, because the sum of the three angles in an oblique triangle equals 180 degrees, we can use two new relationships called the **law of sines** and the **law of cosines** to help solve for unknown angles and sides. These laws are very useful in highway design for determining areas for quantity calculations.

Area of an Oblique Triangle

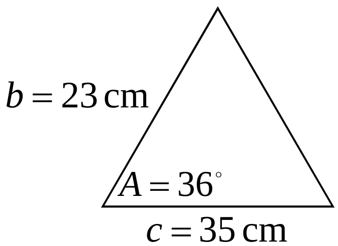
We have a new area formula based on the sine function.

Area of a Triangle $= \frac{1}{2}bc(\sin A)$.



Example:

a. Find the area of the triangle below to the nearest cm.



Area $= \frac{1}{2}bc(\sin A)$. We use the angle in between the two sides and take the sine of that angle.

$$A = \frac{1}{2}bc(\sin 36)$$

$$A = \frac{1}{2}bc(.5878)$$ We then take sides b and c and multiply their lengths together multiplied by one half.

$$A = \frac{1}{2}(23)(35)(.5878) = 236.59 = 237\,\text{cm}^2$$

Law of Sines

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

The capital letters *A,B,C* are all used to denote the angles. The lower case letters *a,b,c* are all used to denote the sides opposite those angles. (For example: for angle *A*, the side opposite it is *a*.)

Apply the law of sines to a triangle if you know:

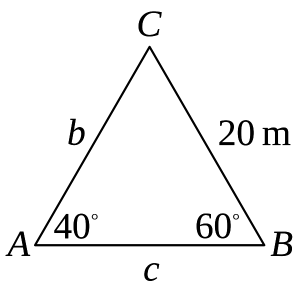
- The measures of two angles and the length of any side,

  or,

- The lengths of two sides and the angle opposite one of the sides.

Example:

a. Solve for the unknown lengths and angles in the triangle below. Round the length of the side to the nearest tenth and each angle to the nearest degree.



We are given the measures of two of the angles so the easiest way to find the third angle is to subtract the two known angles from 180°.
$180^\circ - 40^\circ - 60^\circ = 80^\circ$. Therefore, angle $C = 80$ degrees.

Now, use the law of sines to find the two unknown sides.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \rightarrow \frac{\sin 40^\circ}{20} = \frac{\sin 60^\circ}{b} \rightarrow b(\sin 40°) = 20(\sin 60°)$$

$$b = \frac{20(\sin 60^\circ)}{\sin 40^\circ} = 26.9 \text{ m}$$

$$\frac{\sin A}{a} = \frac{\sin C}{c} \rightarrow \frac{\sin 40^\circ}{20} = \frac{\sin 80^\circ}{c} \rightarrow c(\sin 40°) = 20(\sin 80°)$$

$$c = \frac{20(\sin 80^\circ)}{\sin 40^\circ} = 30.6 \text{ m}$$

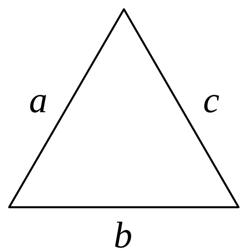
Therefore, $b = 26.9$ m, $c = 30.6$ m, $C = 80$ degrees.

Law of Cosines

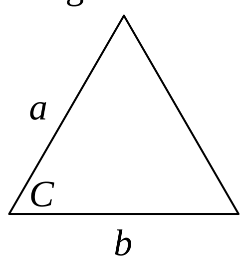
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

You can apply the law of cosines to a triangle if you know:

- The length of 3 sides, <u>or</u>,

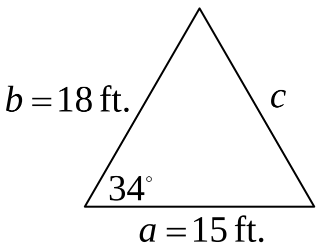


- The length of 2 sides and the included angle.



Examples:

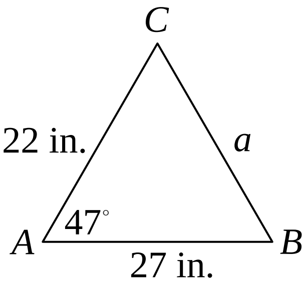
a. Find c in the following triangle to the nearest tenth.



You are given the measure of two sides and the included angle. Use the equation:

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = 15^2 + 18^2 - 2(15)(18)\cos 34^\circ$$
$$c^2 = 101.34$$
$$c = 10.1\text{ ft.}$$

b. Find a in the following triangle to the nearest tenth.



You are given the lengths of two sides and the included angle. Determine a by using the law of cosines. Use the equation $a^2 = b^2 + c^2 - 2bc\cos A$:

$$a^2 = 22^2 + 27^2 - 2(22)(27)\cos 47^\circ$$
$$a^2 = 402.784$$
$$a = 20.1\text{ in.}$$